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LETTER TO THE EDITOR

The statistical properties of the city transport in Cuernavaca (Mexico) and random matrix ensembles

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Abstract. We analyse the statistical properties of the city bus transport in Cuernavaca (Mexico) and show that the subsequent bus arrivals display probability distributions conforming to those given by the unitary ensemble of random matrices.

It is well known that the statistical properties of the coherent chaotic quantum systems are well described by the Wigner/Dyson random matrix ensembles. The fact that the spectral statistics of such chaotic systems is to a large extent generic—a phenomenon known as the universality of quantum chaos—has been confirmed both theoretically and experimentally. (See, for instance, [1] for references.)

The statistical distributions characterizing the ensembles of random matrices can be understood as distributions minimizing the information contained in the system with the constraints that the matrices possess some discrete symmetry properties [2, 3]. Let $P(x_1, x_2, \ldots, x_n)$ denote the joint probability distribution of the eigenvalues $x_1, x_2, \ldots, x_n$ of the given matrix ensemble and

$$I = -\int P(x_1, x_2, \ldots, x_n) \ln(P(x_1, x_2, \ldots, x_n))\, \mathrm{d}x_1, \ldots, \mathrm{d}x_n \tag{1}$$

be its information content. Assuming for instance that the matrices are invariant with respect to the time reversal transformation the information I is minimized when the distribution $P(x_1, x_2, \ldots, x_n)$ conforms to the orthogonal ensemble (GOE) prediction. If there is no external symmetry the total minimum of the information I is achieved for the unitary ensemble (GUE), where the only constrain is that the matrices should be Hermitian.

It has been known for a long time that matrix ensembles are of relevance also for classical one-dimensional interacting many-particle systems, where the matrix eigenvalues $x_1, x_2, \ldots, x_n$ describe the positions of the particles. Therefore the thermal equilibrium of a one-dimensional gas interacting via the Coulomb potential (Dyson gas) has statistical properties (depending on temperature) that are identical with those of the random matrix theory [4]. The same holds true also for other potentials. An example is the Pechukas gas [5] where the one-dimensional particles interact by a potential $\lambda V(x)$ with $V(x) = 1/|x|^2$, x being their mutual distance and λ the relevant coupling constant. Regarding the couplings λ as additive canonical variables it has been shown by Pechukas [5] and Yukawa [6] that the statistical equilibrium of the related canonical ensemble is described by random matrix theory. It has to be stressed, however, that these results were obtained under some special requirements on the dynamics of the variables λ, ensuring in fact a full equivalence of the system to the matrix diagonalization. Nevertheless the methods of statistical physics remain valid also for different shapes of the particle potential as well as for different dynamics of the coupling variables λ. It can be shown that the potential

$$V(x) \approx 1/|x|^a \tag{2}$$

with a being a positive constant, leads also to a random matrix distribution of the particle positions.

The equivalence of the statistical properties of the particle positions of the one-dimensional interacting gas to the random matrix ensemble and the fact that GUE minimizes the information (1) contained in the particle positions lead us to speculate that whenever the information contained in the gas is minimized its statistical properties are described by GUE.

The one-dimensional gas to be studied in this letter is represented by buses that operate city line number four in Cuernavaca (Mexico). We will show that the statistical properties of the bus arrivals are described by the GUE of random matrices. To explain the origin of the interaction between subsequent buses several remarks are necessary. First of all it has to be stressed that there is no covering company responsible for organizing the city transport. Consequently constraints such as a time table that represent external influence on the transport do not exist. Moreover, each bus is the property of the driver. The drivers try to maximize their income and hence the number of passengers they transport. This leads to competition among the drivers and to their mutual interaction. It is known that without additive interaction the probability distribution of the distances between subsequent buses is close to the Poissonian distribution and can be described by the standard bus route model [7]. (In this model the interaction between buses is mediated only through the passengers who randomly arrive on the bus stops and have to be taken by the bus. The rather complicated traffic conditions in the city work as an additive effective randomizer.) A Poisson-like distribution implies, however, that the probability of close encounters of two buses is high (bus clustering) which is in conflict with the effort of the drivers to maximize the number of transported passengers and accordingly to maximize the distance to the preceding bus. In order to avoid the unpleasant clustering effect the bus drivers in Cuernavaca engage people who record the arrival times of buses at significant places. Arriving at a checkpoint, the driver receives the information of when the previous bus passed that place. Knowing the time interval to the preceding bus the driver tries to optimize the distance to it by either slowing down or speeding up. In such a way the obtained information leads to a direct interaction between buses and changes the statistical properties of the related gas.

We have made a record of the arrivals of the buses of line number four close to the city centre. The record contains altogether 3500 arrivals during a time period of 27 days, whereby the arrivals on different days are regarded as being statistically independent. After unfolding the peak times we evaluated the related probability distributions and compared them with the predictions of GUE. In particular we have focused on the bus spacing distribution, i.e. on the probability density $P(s)$ that the time interval between two subsequent buses is equal to s and on the bus number variance $N(T)$ measuring the fluctuations of the total number $n(T)$ of buses arriving at the place during the time interval T:

$$N(T) = \langle (n(T) - T)^2 \rangle \tag{3}$$

where $\langle\ \rangle$ denotes sample averaging. (Note that, after unfolding, the mean interval between buses is equal to one.) According to the prediction of the GUE the spacing distribution and

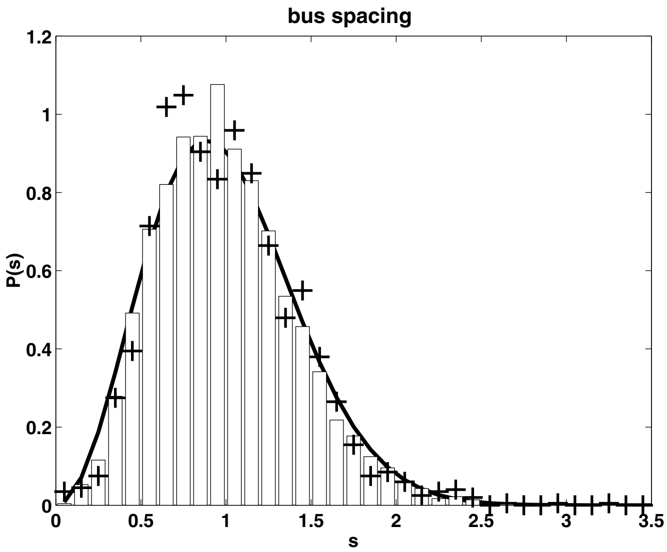

Figure 1. Bus interval distribution $P(s)$ obtained for city line number four. The full curve represents the random matrix prediction (4), the markers (+) represent the bus interval data and bars display the random matrix prediction (4) with 0.8% of the data rejected.

the number variance are given by

$$P(s) = \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right) \tag{4}$$

and

$$N(T) \approx \frac{1}{\pi^2}(\ln 2\pi T + \gamma + 1). \tag{5}$$

These predictions are compared with the obtained bus arrival data and displayed in the following figures.

Figure 1 shows the bus interval distribution when compared with the GUE prediction (4). The bus data are marked by (+). The minor discrepancy between the GUE prediction and the bus data can be explained taking into account the fact that the bus data do not represent the full record. Assuming that roughly 0.8% of the bus arrivals are not notified and rejecting the same number of randomly chosen data from the random matrix eigenvalues, we obtain very satisfactory agreement.

Due to the limited number records available the bus interval distribution is sensitive to the binning used in the evaluation of the probability density $P(s)$. This is why we plot the integrated interval distribution $I(s) = \int P(s')\,\mathrm{d}s'$ that is not liable to binning fluctuations in the next figure. The agreement with the GUE distribution is evident.

The next figure shows the number variance (5) obtained for GUE and compared with the bus data. Here the agreement is good up to time interval $T \approx 3$. For larger T the number variance of the bus arrivals lies significantly above the prediction given by (5). This indicates that the long-range correlations between more than three buses are weaker than predicted by the GUE. The explanation is simple: receiving the time interval information on the preceding bus the driver tries to optimize his position. Doing so he has, however, to take into account also the assumed interval to the bus behind him since otherwise this bus will overtake him.

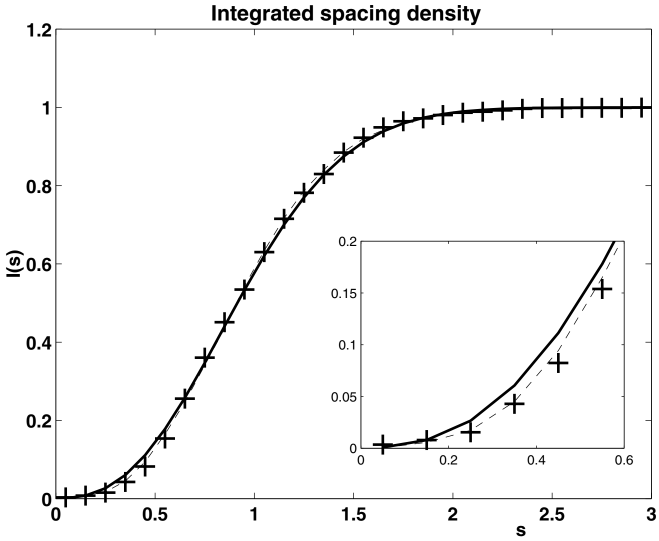

Figure 2. The integrated bus interval distribution $I(s)$ The full curve represents the random matrix prediction, the markers (+) represent the bus data and the dashed line shows the random matrix prediction with 0.8% of the data rejected. The distribution close to the origin is magnified in the inset.

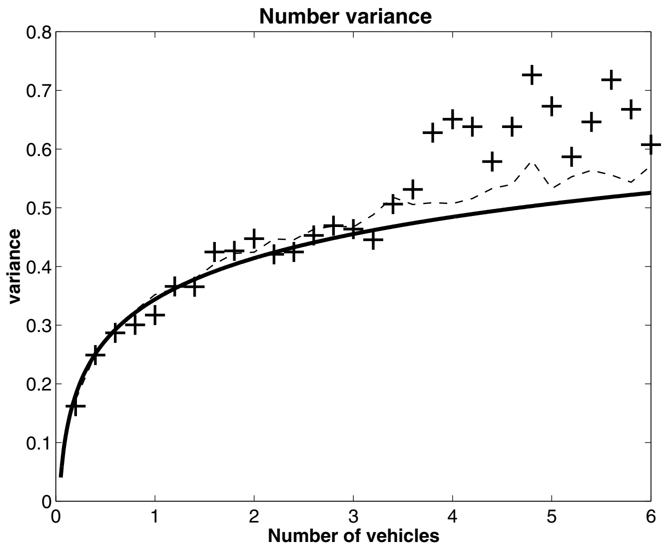

Figure 3. The number variance $N(s)$. The full curve represents the random matrix prediction (5), the markers (+) represent the bus data and the dashed line displays the result obtained with the help of the potential (6) with the summation restricted to three neighbouring particles only.

Hence the driver tries to optimize his position between the preceding and following bus, that leads to the observed correlation.

The GUE properties of the bus arrival statistics can be well understood when regarding the buses as a one-dimensional interacting gas. It has already been mentioned that the exact

GUE statistics is obtained for a Coulomb interaction between the gas particles, i.e., for the interaction potential V given by

$$V = -\sum_{i<j} \log(|x_i - x_j|) + \tfrac{1}{2}\sum_i x_i^2. \tag{6}$$

(In (6) the second terms represent a force confining the gas close to the origin and are not important for our discussion. Equivalently one can discuss the one-dimensional gas moving on a circle instead of a line and then the second term is missing.) The statistical properties of the particle positions of the Dyson gas are identical with those of the random matrix ensembles [8, 9]. In particular the properties of the GUE are recovered by minimizing the information contained in the particle positions.

It is of interest that a similar potential can be indeed found when studying the reaction of drivers to the traffic situation. Here we can use results describing the behaviour of highway drivers. For one-dimensional models it was shown [10, 11] that the ith driver accelerates according to

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} \approx \frac{f(v_{i+1}, v_i)}{(x_{i+1} - xi)^a} \tag{7}$$

where x_{i+1} and v_{i+1} represent the position and velocity of the preceding car respectively, and $f(v_{i+1}, v_i)$ is a function depending on the car velocities only. Approximating f by a constant (justified for low velocities) and taking $a = 1$, we obtain that the cars accelerate in the same way as described by the Coulomb interaction (6).

The exact form of the potential is, however, not crucial for the result. Using the Metropolis algorithm [9] we have numerically evaluated the equilibrium distributions of the positions of a one-dimensional gas interacting via potential (2). When the exponent a is fixed and $a < 2$ the resulting equilibrium distributions belong to the same class as in the Dyson case (6). A similar observation was made also when modelling highway traffic [11], where the results do not qualitatively depend on the exponent a in the acceleration formula (7).

The numerical results show clearly that for a given a one can always find a temperature of the gas such that the equilibrium distribution is given by GUE. Moreover, the fact that the original Dyson potential (6) contains interaction between all pairs of the gas particles is also not substantial. Numerical simulations shows that a good agreement with the random matrix theory is obtained when the summation in (6) is restricted and involves three neighbouring particles only.

The exact interaction between buses in Cuernavaca is not known. However, the weak sensitivity of the statistical equilibrium to the exact form of the potential guides us to the conviction that GUEs are a good choice for the bus description.

We conclude that the statistical properties of the city bus transport in Cuernavaca can be described by a Gaussian GUE of random matrices. This behaviour can be understood as an equilibrium state of an interacting one-dimensional gas under the assumption that the information contained in the positions of the individual gas particles is minimized. The agreement of the recorded bus data with the GUE predictions is surprisingly good.

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